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*Publication date:*  
2013

[Link to publication in Tilburg University Research Portal](#)

*Citation for published version (APA):*

Yanikoglu, I., den Hertog, D., & Kleijnen, J. P. C. (2013). *Adjustable Robust Parameter Design with Unknown Distributions*. (CentER Discussion Paper; Vol. 2013-022). Econometrics.

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No. 2013-022

**ADJUSTABLE ROBUST PARAMETER DESIGN  
WITH UNKNOWN DISTRIBUTIONS**

By

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March 21, 2013

ISSN 0924-7815  
ISSN 2213-9532

# Adjustable Robust Parameter Design with Unknown Distributions

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This article presents a novel combination of robust optimization developed in mathematical programming, and robust parameter design developed in statistical quality control. Robust parameter design uses metamodels estimated from experiments with both controllable and environmental inputs (factors). These experiments may be performed with either real or simulated systems; we focus on simulation experiments. For the environmental inputs, classic robust parameter design assumes known means and covariances, and sometimes even a known distribution. We, however, develop a robust optimization approach that uses only experimental data, so it does not need these classic assumptions. Moreover, we develop ‘adjustable’ robust parameter design which adjusts the values of some or all of the controllable factors after observing the values of some or all of the environmental inputs. We also propose a new decision rule that is suitable for adjustable integer decision variables. We illustrate our novel method through several numerical examples, which demonstrate its effectiveness.

*Key words:* robust optimization; simulation optimization; robust parameter design; phi-divergence

**JEL classification:** C00, C15, C44, C61, C63, C67

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## 1. Introduction

The goal of many *experiments* is to estimate the best solution for a given practical problem. Such experiments may be conducted with a physical system (e.g., an airplane model in a wind tunnel) or a mathematical model of a physical system (e.g., a computerized simulation model of an airplane or an inventory management system). These experiments produce data on the outputs or responses for the given inputs or factors. Output may be univariate (a single or scalar response) or multivariate (multiple responses or a vector with responses). The number of inputs may range from a single input to ‘many’ inputs (e.g., thousands of inputs), but we focus on practical problems with no more than (say) twenty inputs.

*Taguchi* (1987)—for an update see Myers et al. (2009, pp. 483–485)—distinguishes between the following two types of inputs: (i) Controllable or decision factors (say)  $d_j$  ( $j = 1, \dots, k$ ), collected in the  $k$ -dimensional vector  $\mathbf{d} = (d_1, \dots, d_k)^T$ . (ii) Environmental or noise factors (say)  $e_g$  ( $g = 1, \dots, c$ ), collected in the  $c$ -dimensional vector  $\mathbf{e} = (e_1, \dots, e_c)^T$ . By definition, the first type of inputs is under the control of the users; e.g., in an inventory system, management controls the order quantity. The second type of inputs is not controlled by the users; e.g., demand in an inventory system. Taguchi

emphasizes that the noise factors create variability in the outputs. Consequently, the combination of decision factors that (say) maximizes the expected univariate output may cause the variance of that output to be much larger than a combination that ‘nearly’ maximizes the mean output; i.e., a little sacrifice in expected output may save a lot of problems caused by the variability in the output (or as the French proverb states: “the good is the enemy of the best”).

Taguchian *robust parameter design* (RPD) has been criticized by *statisticians*; see the panel discussion reported in Nair et al. (1992). Their main critique concerns the statistical design and analysis in the Taguchian approach; for details we refer to Taguchi (1987) and Myers et al. (2009, pp. 483–495). An alternative RPD approach is given by Myers et al. (2009, pp. 502–506). Like Myers et al. do, we assume that the unknown input/output (I/O) function (say)  $w = g(\mathbf{e}, \mathbf{d})$  for a single output  $w$  of the system is approximated by the following ‘incomplete’ second-order polynomial regression metamodel:

$$y(\mathbf{e}, \mathbf{d}) = \beta_0 + \boldsymbol{\beta}^T \mathbf{d} + \mathbf{d}^T \mathbf{B} \mathbf{d} + \boldsymbol{\gamma}^T \mathbf{e} + \mathbf{d}^T \boldsymbol{\Delta} \mathbf{e} + \epsilon \quad (1)$$

where  $y$  denotes the output of the regression metamodel of the simulation model with output  $w$ ;  $\beta_0$  the intercept;  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)^T$  the first-order effects of the decision variables  $\mathbf{d}$ ;  $\mathbf{B}$  the  $k \times k$  symmetric matrix of second-order effects of  $\mathbf{d}$  with on the main diagonal the ‘purely quadratic’ effects  $\beta_{j;j}$  and off this diagonal half the interactions between pairs of decision factors  $\beta_{j;j'}/2$  ( $j \neq j'$ );  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_c)^T$  the first-order effects of the environmental factors  $\mathbf{e}$ ;  $\boldsymbol{\Delta} = (\delta_{j;g})$  the  $k \times c$  pairwise (two-factor) interactions between  $\mathbf{d}$  and  $\mathbf{e}$ ;  $\epsilon$  the residual with  $E(\epsilon) = 0$  if this model has no ‘lack of fit’ (so it is a valid or adequate approximation of  $g(\mathbf{e}, \mathbf{d})$ ) and with constant variance  $\sigma_\epsilon^2$ .

Myers et al. (2009) assume experiments with real systems; whereas we assume experiments with simulation models of real systems. These simulation models may be either deterministic (especially in engineering) or random (stochastic, possibly with discrete events). Deterministic simulation models have noise if a parameter or input variable has a fixed but unknown value; this is called subjective or epistemic uncertainty; see Iman and Helton (2006). Random simulation models also have objective, aleatory or inherent uncertainty; again see Iman and Helton. We focus on random simulation, but shall also discuss deterministic simulation. Simulation analysts use different names for metamodels, such as response surfaces, surrogates, and emulators; see the many references in Kleijnen (2008, p. 8). There are different types of metamodels, but the most popular types are low-order polynomials such as (1) and Kriging (Gaussian process) models. These polynomials are nonlinear in the inputs  $(\mathbf{e}, \mathbf{d})$  but linear in the regression parameters  $(\beta_0, \dots, \delta_{k;c})$  so the analysis may use classic linear regression models estimated using the least squares (LS) criterion; see Kleijnen (2008, pp. 15–72) and Myers et al. (2009, pp. 13–71). Kriging models are

more flexible so they can accurately approximate the true I/O function over bigger experimental areas; see Kleijnen (2008, pp. 139–156).

Classic RPD assumes that the mean and variance—and sometimes even the probability distribution—of  $\mathbf{e}$  are known. The final parameter design may be sensitive to these assumptions. We therefore propose a robust optimization approach that takes the distribution ambiguity into account, and that uses historical data on the environmental inputs. The developments in *robust optimization* (RO) started with Ben-Tal and Nemirovski (1998, 1999) and El-Ghaoui and Lebret (1997). Optimization problems usually have uncertain coefficients in the objective function and the constraints, so the “nominal” optimal solution—i.e., the optimal solution if there would be no uncertainties—may easily violate the constraints for some realizations of the uncertain coefficients. Therefore, it is better to find a “robust” solution, which is immune to the uncertainty in a so-called uncertainty set. The robust reformulation of a given uncertain mathematical optimization problem is called the *robust counterpart* (RC) problem. The mathematical challenge in RO is to find computationally tractable RCs; see Ben-Tal et al. (2009).

Optimization of systems being simulated—called *simulation optimization* (SO)—is a popular research topic in discrete-event simulation; see Fu (2007), and Fu and Nelson (2003). ‘Robust’ SO, however, is hardly discussed in that literature; only recently, Angün (2011) combines Taguchi’s worldview with *response surface methodology* (RSM). This RSM is a stepwise optimization heuristic that in the various steps uses local first-order polynomial metamodels and in the final step uses a local second-order polynomial metamodel (obviously, these polynomials are linear regression models). Instead of Taguchi’s various criteria, Angün (2011) uses the average value-at-risk (also known as conditional value-at-risk). Miranda and Del Castillo (2011) perform RPD optimization through a well-known SO method; namely, simultaneous perturbation stochastic approximation, which is detailed in Spall (2003). More precisely, Miranda and Del Castillo’s (2011, p. 201) “objective is to determine the operating conditions for a process so that a performance measure  $y$  is as close as possible to the desired target value and the variability around that target is minimum”. Wiedemann (2010, p. 31) applies Taguchian RPD to an agent-based simulation, ensuring that the mean response meets the target value and the variability around that level is sufficiently small. Some years ago, Al-Aomar (2006) used Taguchi’s signal-to-noise ratio and a quality loss function, together with a genetic algorithm with a scalar fitness measure that is a combination of the estimated mean and variance. An older paper is Sanchez (2000), who used Taguchi’s worldview with a loss function that incorporates both system’s mean and variability, and RSM; the author gave many more references. Robust approaches are discussed—albeit briefly—not only in discrete-event simulation but also in deterministic simulation if that simulation has uncertain environmental variables, which (by definition) are beyond management’s control. A recent example is Hu et al. (2012), who apply RO to a

climate simulation model (called ‘DICE’) with input parameters that have a multivariate Gaussian distribution with uncertain or ‘ambiguous’ mean vector and covariance matrix. Another recent example is Dellino et al. (2012), who investigate the well-known economic order quantity model with an uncertain demand rate. Dellino et al. use Taguchi’s worldview, but replace his experimental designs (namely, orthogonal arrays) and metamodels (namely, low-order polynomials) by Latin hypercube sampling and Kriging metamodels. Many more applications can be found in engineering. For example, Chan and Ewins (2010) use Taguchi’s RPD to manage the vibrations in bladed discs. Urval et al. (2010) use Taguchi’s RPD in powder injection moulding of microsystems’ components. Delpiano and Sepulveda (2006) also study RPD in engineering. Vanlı and Del Castillo (2009) study *on-line* RPD (ORPD) to account for control variables that are adjusted over time according to the on-line observations of environmental factors. They assume the posterior predictive densities of responses and environmental factors are known; moreover, they assume that the uncertain parameters follow a specific time series model. They propose two Bayesian approaches for ORPD. In both approaches, controllable factors can be adjusted on-line using an expected loss function and the on-line observations. Joseph (2003) proposes RPD with feed-forward control for measurement systems where true values of the responses cannot be observed but estimated. Using the on-line observable environmental factors, the author proposes a control law that periodically updates the online measurement of the system. Dasgupta and Wu (2006) propose an on-line feedback control mechanism that adjusts the observed output error using a controllable factor—called adjustment factor—that is set on-line during production. All these ORPD methods assume that the mean and variance—sometimes even the probability distribution—of environmental factors are known.

We present a methodology that is a novel combination of RPD and RO. Based on Ben-Tal et al. (2013) and Yanikoğlu and den Hertog (2012), we use experimental data on the simulated system to derive uncertainty sets—to be used in RO—for the unknown probability distribution of the environmental factors. Furthermore, we also use Ben-Tal, den Hertog, and Vial (2012) to convert associated RCs into explicit and computationally tractable optimization problems. Bingham and Nair (2012) point out that “the noise distributions are rarely known, and the choices are often based on convenience”. The advantage of our approach is that it uses the distribution-free structure of RO; i.e., unlike standard approaches we do not make any assumptions about the distribution of the environmental factors such as a Gaussian (or Normal) distribution. Moreover, we develop *adjustable* RPD for those situations in which some or all of the controllable inputs have a ‘wait and see’ character; i.e., their values can be adjusted after observing the values of some or all of the environmental inputs. Examples are the adjustment of several controllable chemical process parameters after observing environmental inputs such as temperature and humidity, or the adjustment of the replenishment order at time  $t$  according to the realized demands in the preceding

$t - 1$  periods in a multistage inventory system. We develop *adjustable* RO for such situations, and show that the corresponding robust counterpart problems can again be reformulated as tractable optimization problems.

The major contributions of our research can be summarized as follows: (i) We propose a RO methodology for a class of RPD problems where the distributional parameters are unknown but historical data on the uncertain parameters are available. (ii) We propose adjustable robust approach for RPD. Unlike classic adjustable RO techniques, our adjustable robust reformulations are tractable even for nonlinear decision rules. (iii) We propose tractable RC formulations of uncertain optimization problems that have quadratic terms in the uncertain parameters. Compared with other studies in the literature, our formulations can handle more general classes of uncertainty sets, and lead to easier tractable formulations. (iv) Last but not least, to the best of our knowledge, our study is the first publication that introduces adjustable integer decision variables in the context of RO, and proposes a specific decision rule for such variables.

We organize this article as follows. §2 summarizes Taguchi's world view and the corresponding regression model formulated in (1), and alternative RPD approaches. §3 proposes our RO approach accounting for distribution ambiguity. §4 develops adjustable RPD. §5 presents numerical examples. §6 summarizes our conclusions, and indicates topics for future research.

## 2. Robust Parameter Design

In this section, we first present the Taguchian RPD approach, and then other popular RPD approaches. To estimate the regression parameters in (1) we need an experiment with the underlying system: Design of experiments (DoE) uses *coded*—also called *standardized* or *scaled*—values (say)  $x_j$  for the factors. So the experiment consists of (say)  $n$  combinations of the coded factors  $\mathbf{x}$ , which correspond with  $\mathbf{d}$  and  $\mathbf{e}$  in (1). Coding is further discussed in Kleijnen (2008, p. 29) and Myers et al. (2009, p. 78).

We reformulate the Taguchian model (1) as the following *linear regression model*:

$$y = \boldsymbol{\zeta}^T \mathbf{x} + \epsilon \quad (2)$$

with the  $\ell$ -dimensional vector of regression parameters or coefficients  $\boldsymbol{\zeta} = (\beta_0, \dots, \delta_{k;c})^T$  and the corresponding vector of regression explanatory variables  $\mathbf{x}$  defined in the obvious way; e.g., the explanatory variable corresponding with the interaction effect  $\beta_{1,2}$  is  $d_1 d_2$ . Then (2) leads to the LS estimator:

$$\hat{\boldsymbol{\zeta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \quad (3)$$

where  $\mathbf{X}$  is the  $n \times \ell$  matrix of explanatory variables with  $n$  denoting the number of scenarios (combinations of control and environmental factors) determined by the DoE that are actually

observed (in a real or simulated system);  $\mathbf{y}$  is the vector with the  $n$  observed outputs. The covariance matrix of the estimator (3) is

$$\text{Cov}(\hat{\boldsymbol{\zeta}}) = \sigma_\epsilon^2 (\mathbf{X}^T \mathbf{X})^{-1} \quad (4)$$

where  $\sigma_\epsilon^2$  was defined in (1). Hence, the variance  $\sigma_\epsilon^2$  may be estimated by the mean squared residuals:

$$\hat{\sigma}_\epsilon^2 = \frac{(\hat{\mathbf{y}} - \mathbf{y})^T (\hat{\mathbf{y}} - \mathbf{y})}{n - \ell} \quad (5)$$

where  $\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\zeta}}$ ; see (2) and (3).

We denote the expected values of the environmental factors through  $E(\mathbf{e}) = \boldsymbol{\mu}_e$ . We allow dependence between the environmental factors so the covariance matrix is  $\text{Cov}(\mathbf{e}) = \boldsymbol{\Omega}_e$ . Analogous to Myers et al. (2009, pp. 504–506), we derive that the metamodel (1) implies that the regression predictor for the mean  $E(y)$  is

$$E_e[y(\mathbf{e}, \mathbf{d})] = \beta_0 + \boldsymbol{\beta}^T \mathbf{d} + \mathbf{d}^T \mathbf{B} \mathbf{d} + \boldsymbol{\gamma}^T \boldsymbol{\mu}_e + \mathbf{d}^T \boldsymbol{\Delta} \boldsymbol{\mu}_e \quad (6)$$

and the regression predictor for the variance  $\text{Var}(y)$  is

$$\text{Var}_e[y(\mathbf{e}, \mathbf{d})] = (\boldsymbol{\gamma}^T + \mathbf{d}^T \boldsymbol{\Delta}) \boldsymbol{\Omega}_e (\boldsymbol{\gamma} + \boldsymbol{\Delta}^T \mathbf{d}) + \sigma_\epsilon^2 \quad (7)$$

where  $(\boldsymbol{\gamma} + \boldsymbol{\Delta}^T \mathbf{d}) = (\partial y / \partial e_1, \dots, \partial y / \partial e_c)^T$  is the gradient with respect to the environmental factors. Obviously, the larger the gradient's elements are, the larger the variance of the predicted output is. Furthermore, if  $\boldsymbol{\Delta} = \mathbf{0}$  (no control-by-noise interactions), then  $\text{Var}(y)$  cannot be controlled through the decision variables  $\mathbf{d}$ . Note the difference between the predicted variance,  $\text{Var}(y)$ , and the variance of the predictor,  $\text{Var}(\hat{y})$  with  $\hat{y} = \mathbf{x}^T \hat{\boldsymbol{\zeta}}$ . Obviously, the mean vector and the covariance matrix completely define a multi-variate Gaussian distribution.

Taguchi focuses his analysis on the *signal-to-noise ratios* (SNRs), which depend on  $E(y) / \sqrt{\text{Var}(y)}$  (the standard deviation  $\sqrt{\text{Var}(y)}$  has the same scale as the mean  $E(y)$ ); see Myers et al. (2009, pp. 486–488). The precise definitions of these SNRs vary with the following goals of RPD: (i) ‘The smaller, the better’: minimize the response. (ii) ‘The larger, the better’: maximize the response. (iii) ‘The target is best’: realize a target value (say)  $T$  for the response.

We do not further dwell on Taguchi’s SNRs, because we think there are better formulations of the various goals of RPD; Myers et al. (2009, pp. 488–495) also question the utility of SNRs. We use the following optimization problem formulation, also formulated by Myers et al. (2009, p. 506):

$$\min_{\mathbf{d}} \text{Var}_e[y(\mathbf{e}, \mathbf{d})] \quad \text{s.t.} \quad E_e[y(\mathbf{e}, \mathbf{d})] \leq T. \quad (8)$$

We may also consider the optimization problem given by Dellino et al. (2012); namely,

$$\min_{\mathbf{d}} E_e[y(\mathbf{e}, \mathbf{d})] \quad \text{s.t.} \quad \text{Var}_e[y(\mathbf{e}, \mathbf{d})] \leq T, \quad (9)$$



or the following variant:

$$\min_{\mathbf{d}} E_{\mathbf{e}}[(y(\mathbf{e}, \mathbf{d}) - T)^2]. \quad (10)$$

**Remark 1** The optimal ‘coded’ controllable factors  $\mathbf{d}^*$ —that minimize the process mean or variance in (8), (9) or (10)—must lie in  $-\mathbf{1} \leq \mathbf{d} \leq \mathbf{1}$ , where  $\mathbf{1}$  denotes the all ones vector. Otherwise, the simulation or physical experiment must be rerun with a larger experimental region for the original (non-coded) input factors to obtain a new response model  $y(\mathbf{e}, \mathbf{d})$  satisfying the associated requirement for the given RPD problem.

In the rest of this paper we focus on (9), since (8) and (10) can be treated analogously.

### 3. Robust Optimization with Unknown Distributions

In this section, we derive the robust reformulations of the class of optimization problems presented in §2. We assume that data on the environmental factors  $\mathbf{e}$  is available or can be obtained via simulation.

#### 3.1. Uncertainty Sets

Instead of relying on the normal distribution, RO derives an *uncertainty set* for the unknown density function of the noise factors. There are several RO approaches, but we follow Ben-Tal et al. (2013) and Yanikoğlu and den Hertog (2012), who develop RO accounting for *historical data* on the noise factors. They do not use a specific distribution to this data; instead, they use the more general concept of *phi-divergence*—as follows.

Given  $N$  historical observations on the noise factors  $\mathbf{e}$ , we construct  $m$  cells such that the number of observations  $o_i$  in cell  $i$  ( $i = 1, \dots, m$ ) is at least five:

$$\sum_{i=1}^m o_i = N \text{ such that } \forall i : o_i \geq 5.$$

The historical data on  $\mathbf{e}$  give the frequencies  $\mathbf{q} = [q_1, \dots, q_m]^T$ , where  $q_i$  is the observed frequency in cell  $i$  so

$$q_i = \frac{o_i}{N}.$$

The phi-divergence measure is

$$I_\phi(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^m q_i \phi\left(\frac{p_i}{q_i}\right)$$

where  $\phi(\cdot)$  satisfy certain mathematical requirements such as  $\phi(1) = 0$ ,  $\phi(a/0) := a \lim_{t \rightarrow \infty} \phi(t)/t$  for  $a > 0$ , and  $\phi(0/0) = 0$ ; for details on phi-divergence we refer to Pardo (2006). It can be proven that the test statistic

$$\frac{2N}{\phi''(1)} I_\phi(\mathbf{p}, \mathbf{q})$$

is asymptotically distributed as a chi-squared random variable with  $(m - 1)$  degrees of freedom. So an asymptotic  $(1 - \alpha)$ -confidence set for  $\mathbf{p}$  is

$$I_\phi(\mathbf{p}, \mathbf{q}) \leq \rho \text{ with } \rho = \rho(N, m, \alpha) = \frac{\phi''(1)}{2N} \chi_{m-1; 1-\alpha}^2. \quad (11)$$

Using (11), Ben-Tal et al. (2013) derive the following *uncertainty set*  $U$  for the unknown probability vector  $\mathbf{p}$ :

$$U = \{\mathbf{p} \in \mathbb{R}^m | \mathbf{p} \geq 0, \sum_{i=1}^m p_i = 1, I_\phi(\mathbf{p}, \mathbf{q}) \leq \rho\}. \quad (12)$$

**Table 1** Phi-Divergence Examples

Divergence	$\phi(t), t > 0$	$I_\phi(\mathbf{p}, \mathbf{q})$	$\phi^*(s)$
Kullback-Leibler	$t \log t$	$\sum_i p_i \log \left( \frac{p_i}{q_i} \right)$	$e^{s-1}$
Burg entropy	$-\log t$	$\sum_i q_i \log \left( \frac{p_i}{q_i} \right)$	$-1 - \log(-s), s \leq 0$
$\chi^2$ -distance	$\frac{1}{t}(t-1)^2$	$\sum_i \frac{(p_i - q_i)^2}{p_i}$	$2 - 2\sqrt{1-s}, s \leq 1$
Pearson $\chi^2$ -distance	$(t-1)^2$	$\sum_i \frac{(p_i - q_i)^2}{q_i}$	$s + s^2/4, s \geq -2$ $-1, s < -2$
Hellinger distance	$(1 - \sqrt{t})^2$	$\sum_i (\sqrt{p_i} - \sqrt{q_i})^2$	$\frac{s}{1-s}, s \leq 1$

Table 1 taken from Ben-Tal et al. (2013) presents common choices of the phi-divergence function;  $\phi^*(s) := \sup_{t \geq 0} \{st - \phi(t)\}$  denotes the conjugate of a phi-divergence distance that will be used for the derivations of RCs in the next section. In various examples, we shall use the  $\chi^2$ -distance in this paper.

### 3.2. Robust Counterparts of Mean and Variance

In this section we derive the robust reformulation of the optimization problem (9), given in §2. Slight modifications of the derivations in this section enable the derivation of the RC of any problem based on mean and variance; e.g., (8) and (10).

We represent each cell  $i$  by its center point  $\mathbf{e}^i$ ; e.g.,  $e_g^i$  is the  $g^{\text{th}}$  ( $g = 1, \dots, c$ ) coordinate of cell  $i$ 's center point. To apply the phi-divergence in (12), we discretize  $y(\mathbf{e}, \mathbf{d})$  by replacing  $\mathbf{e}$  by  $\mathbf{e}^i$ . This substitution gives

$$y_i(\mathbf{d}) = y(\mathbf{e}^i, \mathbf{d}) = \beta_0 + \beta^T \mathbf{d} + \mathbf{d}^T \mathbf{B} \mathbf{d} + (\gamma^T + \mathbf{d}^T \Delta) \mathbf{e}^i. \quad (13)$$

Consequently, the mean of  $y(\mathbf{e}, \mathbf{d})$  becomes approximated by

$$\mathbb{E}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})] = \sum_{i \in V} y_i(\mathbf{d}) p_i = \sum_{i \in V} [\beta_0 + \beta^T \mathbf{d} + \mathbf{d}^T \mathbf{B} \mathbf{d} + (\gamma^T + \mathbf{d}^T \Delta) \mathbf{e}^i] p_i \quad (14)$$

where  $p_i$  denotes the probability of  $\mathbf{e}$  falling into cell  $i$ . Note that  $\mathbf{p}$  is in the uncertainty set  $U$  given by (12) and the empirical estimate  $\mathbf{q}$  of  $\mathbf{p}$  is obtained using the data on  $\mathbf{e}$ ; see §3.1. If we define

$$\psi_i(\mathbf{d}) := (\boldsymbol{\gamma}^T + \mathbf{d}^T \Delta) \mathbf{e}^i, \quad (15)$$

then the variance of  $y(\mathbf{e}, \mathbf{d})$  becomes approximated by

$$\text{Var}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})] = \sum_{i \in V} \psi_i(\mathbf{d})^2 p_i - \left[ \sum_{i \in V} \psi_i(\mathbf{d}) p_i \right]^2. \quad (16)$$

Eventually, the *robust reformulation* of (9) is the following *semi-infinite* optimization problem:

$$\begin{aligned} (\text{SI1}) \quad & \min_{\mathbf{d}} \max_{\mathbf{p} \in U} \sum_{i \in V} [\beta_0 + \boldsymbol{\beta}^T \mathbf{d} + \mathbf{d}^T \mathbf{B} \mathbf{d} + (\boldsymbol{\gamma}^T + \mathbf{d}^T \Delta) \mathbf{e}^i] p_i \\ & \text{s.t.} \quad \sum_{i \in V} \psi_i(\mathbf{d})^2 p_i - \left[ \sum_{i \in V} \psi_i(\mathbf{d}) p_i \right]^2 \leq T \quad \forall \mathbf{p} \in U, \end{aligned} \quad (17)$$

where  $U = \{\mathbf{p} \in \mathbb{R}^m \mid \mathbf{p} \geq 0, \sum_{i=1}^m p_i = 1, I_\phi(\mathbf{p}, \mathbf{q}) \leq \rho\}$ . (SI1) is a difficult optimization problem that has infinitely many constraints (see  $\forall \mathbf{p} \in U$ ), and includes quadratic terms in  $\mathbf{p}$ . Ben-Tal et al. (2009, p. 382) propose a tractable RC of a linear optimization problem with uncertain parameters that appear quadratically, and an ellipsoidal uncertainty. The resulting formulation is a semidefinite programming (SDP) problem; see also Remark 2 below. The following theorem provides tractable RC reformulations of (SI1) for more general ‘phi-divergence’ uncertainty sets.

**Theorem 1** *The vector  $\mathbf{d}$  solves (SI1) if and only if  $\mathbf{d}, \boldsymbol{\lambda}, \boldsymbol{\eta}$ , and  $z$  solve the following RC problem:*

$$\begin{aligned} (\text{RC1}) \quad & \min_{\mathbf{d}, \boldsymbol{\lambda}, \boldsymbol{\eta}, z} \beta_0 + \boldsymbol{\beta}^T \mathbf{d} + \mathbf{d}^T \mathbf{B} \mathbf{d} + \lambda_1 + \rho \eta_1 + \eta_1 \sum_{i \in V} q_i \phi^* \left( \frac{\psi_i(\mathbf{d}) - \lambda_1}{\eta_1} \right) \\ & \text{s.t.} \quad \lambda_2 + \rho \eta_2 + \eta_2 \sum_{i \in V} q_i \phi^* \left( \frac{(\psi_i(\mathbf{d}) + z)^2 - \lambda_2}{\eta_2} \right) \leq T \\ & \quad \eta_1, \eta_2 \geq 0 \end{aligned} \quad (18)$$

where  $\rho$  is given by (11),  $\phi^*(s) := \sup_{t \geq 0} \{st - \phi(t)\}$  denotes the conjugate of  $\phi(\cdot)$ ,  $V = \{1, \dots, m\}$  is the set of cell indices in the uncertainty set  $U$ ,  $q_i$  is the data frequency in cell  $i \in V$  using the historical data on  $\mathbf{e}$ , and  $\boldsymbol{\lambda}, \boldsymbol{\eta}$ , and  $z$  are additional variables.

**Proof.** Using Yanıkoglu and den Hertog (2012, Theorem 1), we can easily derive the explicit RC of the objective function of (SI1) that is linear in  $\mathbf{p} \in U$ . Next we consider the ‘more difficult’ variance constraint (17), which is quadratic in  $\mathbf{p}$ . In the following parts of the proof we use Ben-Tal, den

Hertog, and Vial (2012) to account for the nonlinear uncertainty in the constraint. Using a linear transformation, we reformulate (17) as

$$\max_{\mathbf{a} \in \widehat{U}} g(\mathbf{a}) \leq T, \quad (19)$$

where  $\widehat{U} := \{\mathbf{a} : \mathbf{a} = \mathbf{A}\mathbf{p}, \mathbf{p} \in U\}$ ,  $\mathbf{A}^T = [\psi^2(\mathbf{d}), \psi(\mathbf{d})]$  and  $g(\mathbf{a}) = a_1 - a_2^2$ . Using the indicator function

$$\delta(\mathbf{a}|\widehat{U}) := \begin{cases} 0, & \mathbf{a} \in \widehat{U} \\ +\infty, & \text{elsewhere,} \end{cases}$$

we reformulate (19) as

$$\max_{\mathbf{a} \in \mathbb{R}^2} \{g(\mathbf{a}) - \delta(\mathbf{a}|\widehat{U})\} \leq T. \quad (20)$$

The Fenchel dual of (20)—for details see Rockafellar (1970, pp. 327–341)—is equivalent to

$$\min_{\mathbf{v} \in \mathbb{R}^n} \{\delta^*(\mathbf{v}|\widehat{U}) - g_*(\mathbf{v})\} \leq T \quad (21)$$

where  $\mathbf{v}$  denotes the dual variable, and  $\delta^*(\mathbf{v}|\widehat{U}) := \sup_{\mathbf{p} \in U} \{\mathbf{a}^T \mathbf{v} | \mathbf{a} = \mathbf{A}\mathbf{p}\}$  and  $g_*(\mathbf{v}) := \inf_{\mathbf{a} \in \mathbb{R}^2} \{\mathbf{a}^T \mathbf{v} - g(\mathbf{a})\}$  denote the convex and concave conjugates of the functions  $\delta$  and  $g$ , respectively. Going from (20) to (21) is justified since the intersection of the relative interiors of the domains of  $g(\cdot)$  and  $\delta(\cdot|\widehat{U})$  is non-empty, since  $\mathbf{a} = \mathbf{A}\mathbf{q}$  is always in the relative interiors of both domains. Moreover it is easy to show that  $\delta^*(\mathbf{v}|\widehat{U}) = \delta^*(\mathbf{A}^T \mathbf{v}|U)$ . Then we delete the minimization in (21) because the constraint has the  $\leq$  operator, and the RC reformulation of (17) becomes

$$\delta^*(\mathbf{A}^T \mathbf{v}|U) - g_*(\mathbf{v}) \leq T.$$

Now we derive the complete formulas of the conjugate functions  $\delta^*$  and  $g_*$ . If  $\mathbf{v}^T = [w, z]$ , then the concave conjugate of  $g$  is equivalent to

$$g_*(\mathbf{v}) = \inf_{\mathbf{a} \in \mathbb{R}^2} \{a_1 w + a_2 z - g(\mathbf{a})\} = \begin{cases} -z^2/4, & w = 1 \\ -\infty, & \text{elsewhere.} \end{cases}$$

Using Theorem 1 in Yanikoğlu and den Hertog (2012) once more, the convex conjugate of  $\delta$  is equivalent to

$$\delta^*(\mathbf{A}^T \mathbf{v}|U) = \inf_{\lambda, \eta_2 \geq 0} \left\{ \rho \eta_2 + \lambda + \eta_2 \sum_{i \in V} q_i \phi^* \left( \frac{\psi_i^2(\mathbf{d}) + \psi_i(\mathbf{d})z - \lambda}{\eta_2} \right) \right\}.$$

Thus the RC reformulation of (17) becomes

$$\lambda + \rho \eta_2 + \eta_2 \sum_{i \in V} q_i \phi^* \left( \frac{\psi_i^2(\mathbf{d}) + \psi_i(\mathbf{d})z - \lambda}{\eta_2} \right) + \frac{z^2}{4} \leq T, \eta_2 \geq 0. \quad (22)$$

Substituting  $\lambda_2 = \lambda + z^2/4$  into (22) gives the final RC reformulation (RC1) ■

**Remark 2** An ellipsoidal uncertainty set is a special case of the phi-divergence uncertainty set (12) when the Pearson chi-squared distance is used as the phi-divergence. Moreover, we can reformulate (RC1) as a second order cone problem (SOCP) for the associated distance measure. Notice that SOCP is an ‘easier’ formulation of the problem compared with the SDP by Ben-Tal et al. (2009).

**Remark 3** Ben-Tal et al. (2012) also propose an RC for the variance uncertainty, however the associated RC introduces additional non-convexity. We overcome this difficulty by using the substitution  $\lambda_2 = \lambda + z^2/4$  in the proof of Theorem 1.

We now discuss the ‘general’ computational tractability of (RC1). First,  $\phi^*(h(\mathbf{d}, \lambda_2, z))$  is convex, since the convex conjugate  $\phi^*(s) := \sup_{t \geq 0} \{st - \phi(t)\}$  is non-decreasing in  $s$ , and  $h(\mathbf{d}, \lambda_2, z) = (\psi_i(\mathbf{d}) + z)^2 - \lambda_2$  is convex in  $\mathbf{d}, z$ , and  $\lambda_2$ . It is easy to show that  $\eta_2 \phi^*(\cdot/\eta_2)$  is convex, since the perspective of a convex function is always convex. Eventually, the convexity of the perspective implies that (18) is convex. On the other hand, the objective function of (RC1) is not necessarily convex, since  $y$  is non-convex in  $\mathbf{d}$  unless  $\mathbf{B}$  is a positive semidefinite (PSD) matrix. Nevertheless, (RC1) does not introduce additional non-convexity into the general optimization problem (9).

### 3.3. Alternative Metamodels and Risk Measures

In this subsection, we focus on extensions of our method. §3.3.1 presents a generalization of our method for other metamodels besides (2), and §3.3.2 presents an extension of our method to SNRs. Finally, §3.3.3 shows how to apply our method to tail-risk measures.

**3.3.1. Alternative Metamodels** In this paper we focus on the low-order polynomial (1), since most of the literature and real-life applications use low-order polynomials to approximate the I/O function of the underlying simulation or physical experiment. However, our methodology can also be used to other metamodel types such as higher-order polynomials, Kriging, and radial basis functions. More precisely, consider

$$y(\mathbf{d}, \mathbf{e}) = f(\mathbf{d}) + \psi(\mathbf{d}, \mathbf{e})$$

where  $f(\mathbf{d})$  is the part that affects only the mean of the response (e.g., it is  $f(\mathbf{d}) = \beta_0 + \beta^T \mathbf{d} + \mathbf{d}^T \mathbf{B} \mathbf{d}$  in (1)), and  $\psi(\mathbf{d}, \mathbf{e})$  is the part that affects the response variance (e.g., it is  $\psi(\mathbf{d}, \mathbf{e}) = \gamma^T \mathbf{e} + \mathbf{d}^T \Delta \mathbf{e}$  in (1)). We then reformulate the RC in Theorem 1 as

$$\begin{aligned} \min_{\mathbf{d}, \lambda, \eta, z} \quad & f(\mathbf{d}) + \lambda_1 + \rho \eta_1 + \eta_1 \sum_{i \in V} q_i \phi^* \left( \frac{\psi_i(\mathbf{d}) - \lambda_1}{\eta_1} \right) \\ \text{s.t.} \quad & \lambda_2 + \rho \eta_2 + \eta_2 \sum_{i \in V} q_i \phi^* \left( \frac{(\psi_i(\mathbf{d}) + z)^2 - \lambda_2}{\eta_2} \right) \leq T \\ & \eta_1, \eta_2 \geq 0. \end{aligned}$$

The complexity of  $f(\mathbf{d})$  and  $\psi_i(\mathbf{d}) (= \psi(\mathbf{d}, \mathbf{e}^i))$  determines the complexity of the RC; the problem can be non-convex depending on  $f(\mathbf{d})$  and  $\psi_i(\mathbf{d})$ . Again our robust reformulation does introduce additional variables, but does not introduce additional non-convexity to  $y(\mathbf{e}, \mathbf{d})$ .

**3.3.2. Signal-to-Noise Ratios** SNRs are performance criteria used in many areas including engineering, chemistry, and physics. As we have mentioned earlier in §2, Taguchi (1987) focuses on three performance measures in his SNRs. Our method can be applied for any given function, including the first two measures; namely,  $(y(\mathbf{e}, \mathbf{d}) - 0)^2$  and  $(1/y(\mathbf{e}, \mathbf{d}))^2$ . Therefore, the robust reformulations of the given problems are special cases of Theorem 1, when we have no constraint on the variance. The third measure involves a true SNR that is in line with the following expression:

$$\max_{\mathbf{d}} \frac{\mathbb{E}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})]}{\sqrt{\text{Var}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})]}}, \quad (23)$$

which we reformulate as

$$\max_{\mathbf{d}, w} w \quad \text{s.t.} \quad -\mathbb{E}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})] + w\sqrt{\text{Var}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})]} \leq 0. \quad (24)$$

Like in §3.3.1, we define the response model through  $y(\mathbf{e}, \mathbf{d}) = f(\mathbf{d}) + \psi(\mathbf{e}, \mathbf{d})$ . Using the associated notation, the general RC of (24) after discretization is given by

$$\begin{aligned} (\text{SI2}) \quad & \max_{\mathbf{d}, \lambda, \eta, w, \mathbf{v}} w \\ \text{s.t.} \quad & -f(\mathbf{d}) - \sum_{i \in V} \psi_i(\mathbf{d})p_i + w\sqrt{\sum_{i \in V} \psi_i^2(\mathbf{d})p_i - \left[\sum_{i \in V} \psi_i(\mathbf{d})p_i\right]^2} \leq 0 \quad \forall \mathbf{p} \in U. \end{aligned}$$

The left-hand side of the inequality in (SI2) is concave in  $\mathbf{p}$  when  $w \geq 0$ . The next theorem provides the tractable RC of (SI2).

**Theorem 2** *The vector  $\mathbf{d}$  and  $w$  solve (SI2) if and only if  $\mathbf{d}, \lambda, \eta, w$ , and  $\mathbf{v} = [v_1, v_2]$  solve the following RC constraints:*

$$\begin{aligned} (\text{RC2}) \quad & \max_{\mathbf{d}, \lambda, \eta, w, \mathbf{v}} w \\ \text{s.t.} \quad & -f(\mathbf{d}) + \lambda + \rho\eta + \frac{w^2}{4v_1} + \eta \sum_{i \in V} q_i \phi^* \left( \eta^{-1} \left[ (\psi_i(\mathbf{d})\sqrt{v_1} + v_2)^2 - \psi_i(\mathbf{d}) - \lambda_1 \right] \right) \leq 0, \\ & \eta \geq 0, w \geq 0. \end{aligned}$$

**Proof.** We reformulate the semi-infinite problem (SI2) as

$$\max_{\mathbf{p} \in U} \left\{ -f(\mathbf{d}) - \sum_{i \in V} \psi_i(\mathbf{d})p_i + w\sqrt{\sum_{i \in V} \psi_i^2(\mathbf{d})p_i - \left[\sum_{i \in V} \psi_i(\mathbf{d})p_i\right]^2} \right\} \leq T. \quad (25)$$

Analogous to the proof of Theorem 1, we reformulate (25) as

$$-f(\mathbf{d}) + \max_{\mathbf{a} \in \widehat{U}} g(\mathbf{a}) \leq T, \quad (26)$$

where  $\widehat{U} := \{\mathbf{a} : \mathbf{a} = \mathbf{A}\mathbf{p}, \mathbf{p} \in U\}$ ,  $\mathbf{A}^T = [\psi^2(\mathbf{d}), \psi(\mathbf{d})]$  and  $g(\mathbf{a}) = -a_2 + w\sqrt{a_1 - a_2^2}$ . Using the indicator function

$$\delta(\mathbf{a}|\widehat{U}) := \begin{cases} 0, & \mathbf{a} \in \widehat{U} \\ +\infty, & \text{elsewhere,} \end{cases}$$

we reformulate (26) as

$$-f(\mathbf{d}) + \max_{\mathbf{a} \in \mathbb{R}^2} \{g(\mathbf{a}) - \delta(\mathbf{a}|\widehat{U})\} \leq T.$$

Deleting the minimization in the Fenchel dual of  $\max_{\mathbf{a} \in \mathbb{R}^2} \{g(\mathbf{a}) - \delta(\mathbf{a}|\widehat{U})\}$  as in Theorem 1, the RC is equivalent to

$$-f(\mathbf{d}) + \delta^*(\mathbf{v}|\widehat{U}) - g_*(\mathbf{v}) \leq T \quad (27)$$

where the concave conjugate is

$$g_*(\mathbf{v}) = \inf_{\mathbf{a} \in \mathbb{R}^2} \{a_1 v_1 + a_2 v_2' - g(\mathbf{a})\} = \begin{cases} -[w^2 + (1 + v_2')^2]/4v_1, & v_1 > 0 \\ -\infty, & \text{elsewhere,} \end{cases}$$

and  $\mathbf{v} = [v_1; v_2']$  denotes the additional dual variables. The convex conjugate of  $\delta$  is equivalent to

$$\delta^*(\mathbf{A}^T \mathbf{v}|U) = \inf_{\lambda', \eta \geq 0} \left\{ \rho\eta + \lambda' + \eta \sum_{i \in V} q_i \phi^* \left( \frac{\psi_i^2(\mathbf{d})v_1 + \psi_i(\mathbf{d})v_2' - \lambda'}{\eta} \right) \right\}$$

where  $\eta$  and  $\lambda'$  are the additional Lagrangian dual variables. Thus (27) becomes

$$\begin{aligned} -f(\mathbf{d}) + \rho\eta + \lambda' + \eta \sum_{i \in V} q_i \phi^* \left( \frac{\psi_i^2(\mathbf{d})v_1 + \psi_i(\mathbf{d})v_2' - \lambda'}{\eta} \right) + \frac{w^2 + (1 + v_2')^2}{4v_1} &\leq T \\ \eta \geq 0, w \geq 0. \end{aligned}$$

Using the substitutions  $\lambda = \lambda' + (1 + v_2')^2/4v_1$  and  $v_2 = v_2' + 1/2\sqrt{v_1}$ , the final RC becomes RC2 ■

Notice that when  $v_1$  is fixed, RC2 does not introduce extra non-convexity. We can find the optimal  $v_1$  by solving the problem for various values of  $v_1$ .

**3.3.3. Tail-Risk Constraint** Tail-risk measures are used in finance and engineering to quantify the worst-case risk. In this subsection, we examine the robust reformulation of such constraint given as

$$\mathbb{E}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})] + k\sqrt{\text{Var}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})]} \leq T$$

where  $k \geq 0$ . Analogous to the previous subsection, the general RC is given by

$$(\mathbf{SI3}) \quad f(\mathbf{d}) + \sum_{i \in V} \psi(\mathbf{d})p_i + k\sqrt{\sum_{i \in V} \psi_i(\mathbf{d})^2 p_i - \left[ \sum_{i \in V} \psi_i(\mathbf{d})p_i \right]^2} \leq T \quad \forall \mathbf{p} \in U.$$

**Corollary 1** *The vector  $\mathbf{d}$  solves constraint (SI3) if and only if  $\mathbf{d}, \lambda, \eta$ , and  $\mathbf{v} = [v_1, v_2]$  solve the following RC constraints:*

$$f(\mathbf{d}) + \lambda + \rho\eta + \frac{k^2}{4v_1} + \eta \sum_{i \in V} q_i \phi^* ((\psi_i(\mathbf{d})\sqrt{v_1} + v_2)^2 + \psi_i(\mathbf{d}) - \lambda_1) \leq T,$$

$$\eta \geq 0.$$

**Proof.** The proof follows from Theorem 2, when  $g(\mathbf{a}) = a_2 + k\sqrt{a_1 - a_2^2}$  ■

#### 4. Adjustable Robust Optimization

In the preceding sections the controllable factors  $\mathbf{d}$  are ‘here and now’ decisions; i.e., decisions on  $\mathbf{d}$  must be made before  $\mathbf{e}$  are realized, and hence  $\mathbf{d}$  do not depend on the actual values of  $\mathbf{e}$ . In practice, a part of the controllable factors can often be adjusted after observing the actual values of  $\mathbf{e}$ . For example, in a multi-stage inventory system with uncertain demand, the decisions on the replenishment orders are made one-at-a-time, and the replenishment order  $t$  is placed when the actual demands in periods 1 through  $t-1$  are known. Hence it is realistic to allow the replenishment order for period  $t$  to be adjusted according to the demands in the preceding periods, even though the upcoming demands remain uncertain. The adjustable factors are called ‘wait and see’ decisions. Often here-and-now and wait-and-see decisions appear together in the same problem setting.

To model this situation, Ben-Tal et al. (2009, Chapter 14) reformulate adjustable factors as functions of the uncertain parameters as follows:

$$d_j = D_j(\mathbf{x}_j, \mathbf{P}_j \mathbf{e}) \quad j \in \{1, \dots, n\}, \quad (28)$$

where  $D_j(\cdot)$  are the so-called decision rules that define the class of functions (e.g., affine functions),  $\mathbf{x}_j$  is the vector of coefficient variables to be optimized for the associated function class, and  $\mathbf{P}_j$  is the information-base matrix; e.g., if  $\mathbf{P}_j$  is a zero matrix, then  $\mathbf{d}$  and  $\mathbf{e}$  become functionally independent and we are back to here-and-now decisions; if  $\mathbf{P}_j$  is a unit matrix, then we allow  $d_j$  to depend on all components of  $\mathbf{e}$ . In addition,  $d_j$  can depend on a portion of the observed data on  $\mathbf{e}$ ; e.g., in the multi-stage inventory example,  $\mathbf{P}_j$  has the value 1 in the first  $j-1$  diagonal elements and zero elsewhere.

To obtain the *adjustable robust counterpart* (ARC) of the mean-variance problem (SI1), we replace  $\mathbf{d}$  by  $\mathbf{D}(\mathbf{X}, \mathbf{P}\mathbf{e}) = [D_1(\mathbf{x}_1, \mathbf{P}_1 \mathbf{e}), \dots, D_n(\mathbf{x}_n, \mathbf{P}_n \mathbf{e})]^T$  in the general RC:

$$\min_{\mathbf{X}} \max_{\mathbf{p} \in U} \sum_{i \in V} [\beta_0 + \beta^T \mathbf{D}(\mathbf{X}, \mathbf{P}\mathbf{e}^i) + \mathbf{X}, \mathbf{P}\mathbf{e}^i]^T \mathbf{B} \mathbf{D}(\mathbf{X}, \mathbf{P}\mathbf{e}^i) + (\gamma^T + \mathbf{D}(\mathbf{X}, \mathbf{P}\mathbf{e}^i)^T \Delta) \mathbf{e}^i] p_i$$

$$\text{s.t. } \sum_{i \in V} y_i (\mathbf{D}(\mathbf{X}, \mathbf{P}\mathbf{e}^i))^2 p_i - \left[ \sum_{i \in V} y_i (\mathbf{D}(\mathbf{X}, \mathbf{P}\mathbf{e}^i)) p_i \right]^2 \leq T \quad \forall \mathbf{p} \in U.$$



The tractable ARC of the above problem results from Theorem 1 in §3.2. The adjustable reformulation has the following advantages as we shall detail below. First, the optimal solution of ARC is less conservative than that of the non-adjustable RC, since it is flexible in adjusting the robust optimal decisions according to revealed data. Second, our approach is tractable even for nonlinear decision rules because we translate the uncertainty from  $\mathbf{e}$  to  $\mathbf{p}$ . To point out, in classic *adjustable robust optimization* (ARO) tractability is generally scarce for non-linear decision rules; Ben-Tal et al. (2009, pp. 382-387) show that the explicit ARC for a quadratic decision rule is derivable only for an ellipsoidal uncertainty set, and the resulting ARC is an SDP problem. Third, we propose a new decision rule that enables modeling integer adjustable decision variables, and this is completely new in the RO literature.

To present the associated decision rules we use the following illustrative example:

$$\min_{\mathbf{d}} \mathbb{E}_{\mathbf{e}} \left[ (1 + 5d_1 + 5d_2 + e_1 - e_2)^2 + (1 + 5d_1 + 10d_2 + e_1 + e_2)^2 \right] \quad (29)$$

where all factors are coded. We divide the support of  $\mathbf{e}$ —namely, the unit box in two dimension  $[-1, 1]^2$ —into four cells of equal size. Hence the center points of the cells are  $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4\} = \{(-0.5, -0.5), (-0.5, 0.5), (0.5, -0.5), (0.5, 0.5)\}$ . So the nominal problem after discretization is

$$\min_{\mathbf{d}} \sum_{i=1}^4 \left[ (1 + 5d_1 + 5d_2 + e_1^i - e_2^i)^2 + (1 + 5d_1 + 10d_2 + e_1^i + e_2^i)^2 \right] q_i. \quad (30)$$

Suppose the observed data are  $\mathbf{q} = [0.4, 0.3, 0.2, 0.1]^T$ . It is easy to derive the optimal solution of this *nominal* problem:  $(d_1, d_2) = (-0.08, -0.08)$ . The uncertainty set for the RC is given by

$$\mathcal{P} := \left\{ \mathbf{p} = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4 \mid p_1 + p_2 + p_3 + p_4 = 1, \sum_{i=1}^4 \frac{(p_i - q_i)^2}{p_i} \leq 0.5, \mathbf{p} \geq 0 \right\}.$$

Using this uncertainty set, we derive that the worst-case objective value for the nominal solution is 1.2; moreover, the non-adjustable robust counterpart of (29) is

$$\min_{\mathbf{d}} \max_{\mathbf{p} \in \mathcal{P}} \sum_{i=1}^4 \left[ (1 + 5d_1 + 5d_2 + e_1^i - e_2^i)^2 + (1 + 5d_1 + 10d_2 + e_1^i + e_2^i)^2 \right] p_i,$$

and its robust optimal solution is  $(-0.2, 0)$  with objective value 1.0. Next we shall examine three adjustable formulations; namely, linear, nonlinear, and cell-based decision rules.

**Linear Decision Rule** Now we assume that the functions of the adjustable controllable factors  $\mathbf{d}$  are linear in the observed values of the environmental factors  $\mathbf{e}$ . The *fully adjustable* linear decision rule is

$$d_j = D_j(\mathbf{x}_j, \mathbf{I}_2 \mathbf{e}) := x_{j0} + x_{j1}e_1 + x_{j2}e_2 \quad \forall j \in \{1, 2\} \quad (31)$$

where  $I_2 = [1, 0; 0, 1]$ . When both controllable factors are fully adjustable, all decisions are made only after the uncertainty is revealed. Consequently, the ARC of (29) with linear decision rule is

$$\min_{\mathbf{X}} \max_{\mathbf{p} \in \mathcal{P}} \sum_{i=1}^4 \left[ \left( 1 + 5D_1(\mathbf{x}_1, \mathbf{e}^i) + 5D_2(\mathbf{x}_2, \mathbf{e}^i) + e_1^i - e_2^i \right)^2 + \left( 1 + 5D_1(\mathbf{x}_1, \mathbf{e}^i) + 10D_2(\mathbf{x}_2, \mathbf{e}^i) + e_1^i + e_2^i \right)^2 \right] p_i.$$

The optimal solution is  $\mathbf{x}_1^* = (-0.2, -0.2, 0.6)^T$  and  $\mathbf{x}_2^* = (0, 0, -0.4)^T$ ; i.e., the linear decision rules are  $D_1(\mathbf{x}_1, \mathbf{e}) = -(1 + e_1 - 3e_2)/5$  and  $D_2(\mathbf{x}_2, \mathbf{e}) = -2e_2/5$ , so we have

$$\begin{aligned} 5d_1 + 5d_2 &= 5D_1(\mathbf{x}_1, \mathbf{e}) + 5D_2(\mathbf{x}_2, \mathbf{e}) = -1 - e_1 + e_2 \\ 5d_1 + 10d_2 &= 5D_1(\mathbf{x}_1, \mathbf{e}) + 10D_2(\mathbf{x}_2, \mathbf{e}) = -1 - e_1 - e_2. \end{aligned}$$

Therefore, the ARC yields the lowest possible optimal objective value; namely, zero for the problem, whereas the non-adjustable RC yields one.

**Table 2** Linear Decision Rules (LDR) and Objective Values

LDR		$D_1(\cdot)$	$D_2(\cdot)$	Obj.
$d_1$	$d_2$	$(x_{10}, x_{11}, x_{12})$	$(x_{20}, x_{21}, x_{22})$	
na	na	$(-0.2, -, -)$	$(0, -, -)$	1.00
na	e1	$(-0.189, -, -)$	$(-0.009, -0.103, -)$	0.66
na	e2	$(-0.2, -, -)$	$(0, -, 0)$	1.00
e1	na	$(-0.2, -0.2, -)$	$(0, -, -)$	0.50
e2	na	$(-0.2, -, 0)$	$(0, -, -)$	1.00
na	e[1,2]	$(-0.186, -, -)$	$(-0.008, -0.12, -0.04)$	0.62
e[1,2]	na	$(-0.2, -0.2, 0)$	$(0, -, -)$	0.50
e1	e1	$(-0.2, -0.2, -)$	$(0, -, -)$	0.50
e1	e2	$(-0.2, -0.2, 0)$	$(0, -, -0.04)$	0.45
e2	e1	$(-0.21, -, 0.04)$	$(-0.001, -0.09, -)$	0.68
e2	e2	$(-0.2, -, 0.6)$	$(0, -, -0.4)$	0.50
e[1,2]	e1	$(-0.2, -0.2, 0)$	$(0, -, -)$	0.50
e[1,2]	e2	$(-0.2, -0.2, 0.6)$	$(-, -, 0.4)$	0.00
e1,	e[1,2]	$(-0.2, -0.2, -)$	$(0, 0, -0.04)$	0.45
e2,	e[1,2]	$(-0.2, -, 0.6)$	$(0, -0.12, -0.4)$	0.05
e[1,2]	e[1,2]	$(-0.2, -0.2, 0.6)$	$(0, 0, -0.4)$	0.00

More interesting cases have wait-and-see and here-and-now decisions together or at least one of the controllable factors is not fully adjustable. Table 2 presents the numerical results for all possible combinations of linear decision rules. The first column gives the possible linear decision rules for the two adjustable factors  $d_1$  and  $d_2$ , where ‘na’ denotes *non-adjustable*, ‘e1’ denotes a factor that is adjustable on  $e_1$ ; similarly, ‘e2’ denotes adjustability on  $e_2$ , and ‘e[1,2]’ denotes a

fully adjustable factor. The second and third columns are the optimal coefficients (variables)  $\mathbf{x}_1$  and  $\mathbf{x}_2$  of the decision rules  $D_1(\cdot)$  and  $D_2(\cdot)$ , where  $(-)$  denotes a variable that vanishes in the associated decision rule. The final column (Obj.) presents the robust optimal objective value for the associated decision rule. Altogether, the numerical results show that when one of the factors is non-adjustable and the other is adjustable on  $e_2$ —see row (na, e2) or (e2, na) in Table 2—the optimal objective value of the ARC is the same as that of the non-adjustable RC. In all other cases the optimal objective value of the non-adjustable RC improves with at least 32% (see row (e2, e1)) for the ARC, and the highest improvement (100%) is attained when the first factor is fully adjustable and the second one is non-adjustable; see row (e[1,2], e2). Another interesting outcome is that introducing an adjustable factor into the problem may change the optimal decision for the non-adjustable factor; i.e., an optimal here-and-now factor can have different values in the ARC and RC. For example, if  $d_1$  is adjustable on  $e_1$  and  $d_2$  is non-adjustable, then the optimal  $d_2$  is -0.189 in the ARC, but it is -0.2 in the RC; see (na, na) and (na, e1).

**Nonlinear Decision Rule** Table 3 shows the following nonlinear (quadratic) decision rule:

$$D_j(\mathbf{x}_j, \mathbf{I}_2 \mathbf{e}) := x_{j0} + x_{j1}e_1 + x_{j2}e_2 + x_{11}^{(j)}e_1^2 + x_{22}^{(j)}e_2^2 + x_{12}^{(j)}e_1e_2 \quad (32)$$

where  $\mathbf{x}_j = [x_{j0}, x_{j1}, x_{j2}, x_{11}^{(j)}, x_{12}^{(j)}, x_{22}^{(j)}]$ . For example, if the controllable factors are only adjustable in  $e_1$ , then the decision rule is

$$D_j(\mathbf{x}_j, P_j \mathbf{e}) := x_{j0} + x_{j1}e_1 + x_{11}^{(j)}e_1^2$$

where  $P_j = [1, 0; 0, 0]$ . Obviously, the nonlinear decision rule is more general than the linear rule, so it is at least as good as the linear decision rule used in Table 2.

We denote the cases where the nonlinear decision rule performs better than the linear by (\*) in the last column of Table 3. The highest improvement compared with Table 2 is obtained when the first factor is non-adjustable and the second factor is fully adjustable; compare (na, e[1,2]) in Table 3, where the optimal objective value is 0.584, with the objective 0.621 for the same situation in Table 2. Quantifying the *value of information* is an important topic in both adjustable robust decision making and in general decision making; Table 3 shows that having extra information on  $e_2$  but not on  $e_1$  for one of the controllable factors has no added value in the adjustable decision making; i.e., the non-adjustable and the adjustable RCs have the same optimal objective at (na, e2) and (e2, na). On the other hand, having information on  $e_1$  for one of the controllable factors yields improvement in the objective; see (na, e1) and (e1, na) in Table 3. Moreover, if  $d_1$  responds to both environmental factors, and  $d_2$  uses information on  $e_2$  only, then we obtain the lowest possible optimal objective value (namely, zero); see (e[1,2], e2) in Table 3.

**Table 3** Nonlinear Decision Rules (NDR) and Objective Values

NDR		$D_1(\cdot)$	$D_2(\cdot)$	Obj.
$d_1$	$d_2$	$(x_{10}, x_{11}, x_{12}, x_{11}^{(1)}, x_{22}^{(1)}, x_{12}^{(1)})$	$(x_{20}, x_{21}, x_{22}, x_{11}^{(2)}, x_{22}^{(2)}, x_{12}^{(2)})$	
na	na	(-0.2, -, -, -, -)	(0, -, -, -, -)	1.00
na	e1	(-0.196, -, -, -, -)	(0.014, 0.093, -, -0.079, -, -)	0.65*
na	e2	(-0.2, -, -, -, -)	(-0.33, -, 0, -, 1.33, -)	1.00
e1	na	(-0.195, -0.2, -, -0.02, -, -)	(0, -, -, -, -)	0.50
e2	na	(-0.192, -, 0, -, -, -)	(-, -0.03, -, -, -, -)	1.00
na	e[1,2]	(-0.188, -, -, -, -)	(0.062, -0.106, -0.053, -0.244, 0.015, 0.05)	0.58*
e[1,2]	na	(-0.178, -0.2, 0, -0.044, -0.044, 0)	(0, -, -, -, -)	0.50
e1	e1	(-0.213, -0.2, -, 0.05, -, -)	(-0.002, 0, -, 0.008, -, -)	0.50
e1	e2	(-0.533, -0.2, -, 1.33, -, -)	(-0.005, -, -0.04, -, 0.02, -)	0.45
e2	e1	(-0.19, -, 0.044, -, -0.047, -)	(-0.045, -0.063, -, 0.197, -, -)	0.65*
e2	e2	(-0.223, -, 0.6, -, 0.094, -)	(-0.002, -, -0.4, -0.008, -)	0.50
e[1,2]	e1	(-0.186, -0.2, 0, -0.028, 0.028, 0)	(-0.006, -, -, 0.023, -, -)	0.50
e[1,2]	e2	(-0.253, -0.2, 0.6, 0.107, 0.107, 0)	(-0.002, -, -0.4, -, 0.008, -)	0.00
e1	e[1,2]	(-0.146, -0.199, -, -0.214, -, -)	(-0.052, -0.003, -0.047, 0.148, 0.057, -0.001)	0.44*
e2	e[1,2]	(-0.233, 0.6, -, -0.133, 0)	(-0.151, -0.12, -0.4, 0.105, 0.5, -)	0.05
e[1,2]	e[1,2]	(-0.214, -0.2, 0.6, 0.028, 0.028, 0)	(0.013, 0, -0.4, -0.066, 0.014, 0)	0.00

(\*) denotes an improved optimal objective value compared with that in Table 2

**Cell-Based Decision Rule** Now we propose a new type of decision rule that we call the *cell-based* decision rule:

$$D_j(\mathbf{x}_j, \mathbf{e}) := x_{ji} \text{ if } \mathbf{e} \in \text{cell}(i), i \in V_j, \quad (33)$$

where  $\mathbf{x}_j$  is the decision vector for the  $j$ th adjustable variable ( $x_{ji}$  being the decision for the  $i$ th cell),  $\text{cell}(i)$  is the region determined by the  $i$ th cell, and  $V_j$  is the set of cell indices for the associated information-base.

**Remark 4** Cells used in the decision rule are non-intersecting—excluding the boundaries—squares in two dimensions, cubes in three dimensions, and hypercubes in higher dimensional uncertainty spaces. For the sake of simplicity, we assume that—given the dimension—all cells have the same volume.

If in the illustrative example (29),  $d_1$  is fully adjustable, then the decision rule is

$$D_1(\mathbf{x}_1, \mathbf{e}) := \begin{cases} x_{11}, & \mathbf{e} \in \text{cell}(1) \\ x_{12}, & \mathbf{e} \in \text{cell}(2) \\ x_{13}, & \mathbf{e} \in \text{cell}(3) \\ x_{14}, & \mathbf{e} \in \text{cell}(4) \end{cases} \quad (34)$$

where  $\text{cell}(i) := \{(e_1, e_2) \in \mathbb{R}^2 : \ell_{1i} \leq e_1 \leq u_{1i}, \ell_{2i} \leq e_2 \leq u_{2i}\}$  ( $i \in \{1, 2, 3, 4\}$ ),  $\ell_1 = [0, -1, -1, 0]$ ,  $\mathbf{u}_1 = [1, 0, 0, 1]$ ,  $\ell_2 = [0, 0, -1, -1]$ ,  $\mathbf{u}_2 = [1, 1, 0, 0]$ , and  $V_1 = \{1, 2, 3, 4\}$ ; cells are represented by their

center points in (30). To show the difference between full and partial information, we assume that  $d_1$  is adjustable on  $e_1$  but not on  $e_2$ . The associated decision then becomes

$$D_1(\mathbf{x}_1, e_1) := \begin{cases} x_{11}, & e_1 \in \text{cell}(1) \\ x_{12}, & e_1 \in \text{cell}(2) \end{cases} \quad (35)$$

where  $\text{cell}(1) := \{e_1 \in \mathbb{R} : 0 \leq e_1 \leq 1\}$  and  $\text{cell}(2) := \{e_1 \in \mathbb{R} : -1 \leq e_1 \leq 0\}$ , and  $V_1 = \{1, 2\}$ . It is easy to see that (35) implies that when  $e_2$  is extracted from the information-base, the new cells are projections from the cells in the two-dimensional space in (34) onto the one-dimensional space on  $e_1$  in (35). The disadvantage of the cell-based decision rule is that this rule often has more variables compared with the linear and nonlinear decision rules, especially when the number of cells is high. Nevertheless the numerical results for the example show that the new decision rule is better than the linear, and is ‘almost’ as good as the nonlinear decision rule—even when the total number of cells is only four; see Table 4.

**Table 4** Cell-Based Decision Rules (CDR) and Objective Values

CDR		$D_1(\cdot)$	$D_2(\cdot)$	Obj.
$d_1$	$d_2$	$(x_{11}, x_{12}, x_{13}, x_{14})$	$(x_{21}, x_{22}, x_{23}, x_{24})$	
na	na	(-0.2, -, -, -)	(0, -, -, -)	1.00
na	e1	(-0.19, -, -, -)	(-0.06, 0.04, -, -)	0.66
na	e2	(-0.2, -, -, -)	(0, 0, -, -)	1.00
e1	na	(-0.3, -0.1, -, -)	(0, -, -, -)	0.50
e2	na	(-0.2, -0.2, -, -)	(0, -, -, -)	1.00
na	e[1,2]	(-0.19, -, -, -)	(-0.09, 0.03, 0.07, -0.05)	0.62
e[1,2]	na	(-0.3, -0.1, -0.1, -0.3)	(0, -, -, -)	0.50
e1	e1	(-0.3, -0.1, -, -)	(0, 0, -, -)	0.50
e1	e2	(-0.3, -0.1, -, -)	(-0.02, 0.02, -, -)	0.45
e2	e1	(-0.18, -0.2, -, -)	(-0.06, 0.04, -, -)	0.65
e2	e2	(0.1, -0.5, -, -)	(-0.2, 0.2, -, -)	0.50
e[1,2]	e1	(-0.3, -0.1, -0.1, -0.3)	(0, 0, -, -)	0.50
e[1,2]	e2	(0, 0.2, -0.4, -0.6)	(-0.2, 0.2, -, -)	0.00
e1	e[1,2]	(-0.3, -0.1, -, -)	(-0.02, -0.02, 0.02, 0.02)	0.45
e2	e[1,2]	(0.1, -0.5, -, -)	(-0.26, -0.14, 0.26, 0.14)	0.05
e[1,2]	e[1,2]	(0, 0.2, -0.4, -0.6)	(-0.2, -0.2, 0.2, 0.2)	0.00

To the best of our knowledge, decision rules in the RO literature cannot handle adjustable *integer* variables, since the adjustable decision is a function of the uncertain parameter  $\mathbf{e}$ , and the function does not necessarily take integer values for all  $\mathbf{e}$ ; see (31) and (32). However, our cell-based decision rule can handle such variables. As we can see from (33), the adjustable decision  $x_{ji}$  can take integer values since the cell-based decision rule relates  $\mathbf{e}$  and  $x_{ij}$  through an ‘if’ statement. Therefore, if we make  $x_{ij}$  an integer variable, then the cell-based decision rule gives integer decisions. Using the illustrative example, we show the validity of our approach for such a problem. We modify the old

example in the following way: We assume  $d_1$  and  $d_2$  are adjustable, and they take values that are multiples of  $1/4$ .

**Table 5** Integer Cell-Based Decision Rules and Objective Values

CDR		$D_1(\cdot)$	$D_2(\cdot)$	Obj.
$d_1$	$d_2$	$(x_{11}, x_{12}, x_{13}, x_{14})$	$(x_{21}, x_{22}, x_{23}, x_{24})$	
na	na	$(-0.25, -, -, -)$	$(0, -, -, -)$	1.41
na	e1	$(-0.25, -, -, -)$	$(0, 0, -, -)$	1.41
na	e2	$(-0.25, -, -, -)$	$(0, 0, -, -)$	1.41
e1	na	$(-0.25, 0, -, -)$	$(0, -, -, -)$	0.92
e2	na	$(-0.25, -0.25, -, -)$	$(0, -, -, -)$	1.41
na	e[1,2]	$(-0.25, -, -, -)$	$(0, 0, 0, 0)$	1.41
e[1,2]	na	$(-0.25, 0, 0, -0.25)$	$(0, -, -, -)$	0.92
e1	e1	$(-0.25, 0, -, -)$	$(0, 0, -, -)$	0.92
e1	e2	$(-0.25, 0, -, -)$	$(0, 0, -, -)$	0.92
e2	e1	$(-0.25, -0.25, -, -)$	$(0, 0, -, -)$	1.41
e2	e2	$(0.25, -0.5, -, -)$	$(-0.25, 0.25, -, -)$	1.29
e[1,2]	e1	$(-0.25, 0, 0, -0.25)$	$(0, 0, -, -)$	0.92
e[1,2]	e2	$(0, 0.25, -0.50, -0.75)$	$(-0.25, 0.25, -, -)$	0.26
e1	e[1,2]	$(-0.25, 0, -, -)$	$(0, 0, 0, 0)$	0.92
e2	e[1,2]	$(0, -0.50, -, -)$	$(-0.25, 0, 0.25, 0.25)$	0.99
e[1,2]	e[1,2]	$(0, 0.25, -0.50, -0.75)$	$(-0.25, -0.25, 0.25, 0.25)$	0.26

Table 5 presents optimal decision rules and resulting objective values for all possible combinations of adjustability. These numerical results show that we obtain important improvements for the non-adjustable optimal objective value by using the cell-based decision rule. As may be anticipated, the integer formulation yields higher (worse) optimal objective values. For example, we can no longer get a zero objective in Table 5. Moreover, in contrast to the continuous case, we can no longer improve the optimal objective of the non-adjustable RC at decision rule combinations: (na, e1), (na, e[1,2]), and (e2, e1); see the corresponding rows of Tables 4 and 5.

## 5. Realistic Examples

In this section, we present realistic examples to demonstrate the effectiveness of our methods. We use a 64-bit Windows PC with a 2.2 GHz Intel Core i7 processor, and 8 GB of RAM. To solve the mathematical optimization problems, we use KNITRO 8.0 embedded in MATLAB (2012a) and AIMMS 3.12.

### 5.1. Television Images

In color televisions the quality of signals is determined by the power signal-to-noise-ratios in the image transmitted. We take the response function  $y(\mathbf{d}, \mathbf{e})$  from Myers et al. (2009, p. 512), where

the response  $y$  measures the quality of transmitted signals in decibels. The controllable factors are the number of tabs in a filter  $d_1$ , and the sampling frequency  $d_2$ ; the environmental factors are the number of bits in an image  $e_1$ , and the voltage applied  $e_2$ . The least-square estimate of the metamodel is

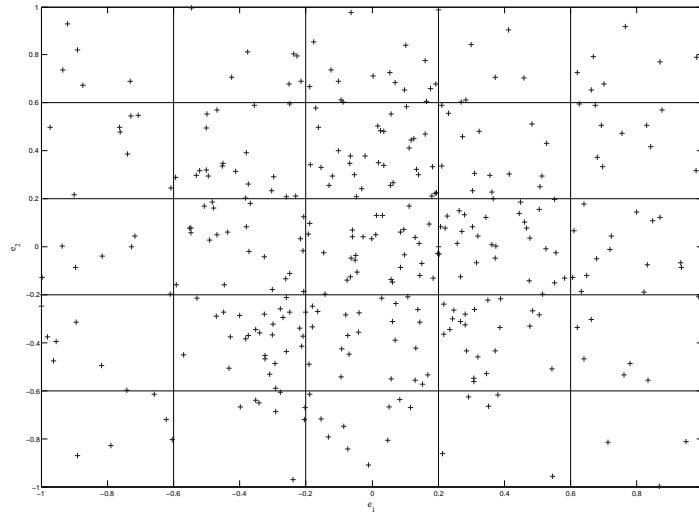
$$\begin{aligned}\hat{y}(\mathbf{d}, \mathbf{e}) = & 33.389 - 4.175d_1 + 3.748d_2 + 3.348d_1d_2 - 2.328d_1^2 - 1.867d_2^2 \\ & - 4.076e_1 + 2.985e_2 - 2.324d_1e_1 + 1.932d_1e_2 + 3.268d_2e_1 - 2.073d_2e_2\end{aligned}$$

where all factors are coded; for details on the DoE we refer to Myers et al. (2009, pp. 511–515).

We find the optimal design settings of  $d_1$  and  $d_2$  using the optimization problem (9):

$$\begin{aligned}\max_{\mathbf{d}} \quad & \mathbb{E}_{\mathbf{e}}[\hat{y}(\mathbf{d}, \mathbf{e})] \\ \text{s.t.} \quad & \text{Var}_{\mathbf{e}}[\hat{y}(\mathbf{d}, \mathbf{e})] \leq T.\end{aligned}\tag{36}$$

The robust counterpart of (36) is given by Theorem 1 in §3.2. To estimate  $\mathbf{q}$ , we use the historical data in Figure 1. Since we have no real data, we have randomly created these data. The sample size is  $N = 350$ , and the support of  $\mathbf{e}$  is divided into 25 cells of the same volume so  $V = \{1, \dots, 25\}$ ,  $\mathbf{q} = [q_1, \dots, q_{25}]^T$  and  $\rho = \chi_{0.999, 24}^2 / 350$ ; see (11). We shall use the same data in our two realistic examples so the data do not favor our method.



**Figure 1** Historical Data on  $\mathbf{e}$

The goal of these examples is to compare the optimal solutions of the nominal and robust counterpart problems. In §5.1.1 we shall compare the worst-case and average performances of these two solutions via the objective value and the constraint violation. In §5.1.2 we shall compare the confidence levels probabilities of the nominal and robust optimal solutions.

**5.1.1. Robust versus Nominal Solutions** We vary the right-hand side value  $T$  of the variance constraint from 0.1 to 0.8; see column one in Table 6. We solve the nominal and the RC problems for these  $T$  values, and compare the worst-case performances of the nominal and robust optimal solutions.

**Table 6** Worst-Case Analysis for TV Image Example

$T$	Robust			Nominal		
	$(d_1^*, d_2^*)$	$y^*$	Var.(%)	$(d_1^*, d_2^*)$	W-C( $\mathbb{E}$ )	W-C(Var)(%)
0.1	(-0.43,0.83)	35.27	0.10(0%)	(-0.4472,0.7755)	35.27	0.19(88%)
0.2	(-0.44,0.79)	35.28	0.15(-25%)	(-0.4625,0.6853)	35.26	0.37(87%)
0.3	(-0.44,0.79)	35.28	0.15(-50%)	(-0.4763,0.6152)	35.23	0.56(86%)
0.4	(-0.44,0.79)	35.28	0.15(-62%)	(-0.4867,0.5648)	35.20	0.71(79%)
0.5	(-0.44,0.79)	35.28	0.15(-70%)	(-0.4867,0.5648)	35.20	0.71(43%)
0.6	(-0.44,0.79)	35.28	0.15(-75%)	(-0.4867,0.5648)	35.20	0.71(19%)
0.7	(-0.44,0.79)	35.28	0.15(-78%)	(-0.4867,0.5648)	35.20	0.71( 2%)
0.8	(-0.44,0.79)	35.28	0.15(-81%)	(-0.4867,0.5648)	35.20	0.71(-11%)

Columns two and three are the robust optimal solution  $(d_1^*, d_2^*)$  and its objective value  $(y^*)$ . Column four (Var.) is the robust variance of the response. Column five is the nominal optimum solution. Column six presents the mean ( $\mathbb{E}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})]$ ) when the decision factors are fixed to the nominal optimum solution  $\mathbf{d}^*$  and the worst-case probability vector  $\mathbf{p}^*$  that minimizes the expectation is realized. Column seven (W-C.(Var)) gives the worst-case value of the variance for the nominal solution; now we consider the probability vector  $\mathbf{p}^*$  that maximizes the variance, as the worst-case. Notice that we also report the percentages of the worst-case constraint violations for the nominal and robust optimal solutions; these values are shown within brackets (%) in the columns four and seven. We use the formula  $[(\text{W-C.}(\text{Var})-T) \times 100/T]$  to calculate the constraint violation percentage of the nominal solution when the worst-case uncertain parameters are realized. If the violation percentage is negative, then the constraint is satisfied; 0% means the constraint is binding (see column four, row  $T=0.1$ ); a positive percentage means constraint violation at column seven.

Table 6 reveals that the robust reformulation of the variance constraint becomes redundant when  $T \geq 0.2$ ; e.g., the left-hand side value of the constraint for the robust optimal solution  $(-0.4439, 0.7988)$  when  $T = 0.2$  is 0.149. Therefore we have the same robust optimal solution for  $T \geq 0.2$ . The nominal variance constraint (not the worst-case) also becomes redundant when  $T \geq 0.389$ , and therefore we have the same nominal optimum solution when  $T \geq 0.4$ . Notice that the robust optimal objective values and the worst-case objective values of the nominal optimum solutions are within 1% of each other. However, the constraint violation percentages favor our robust approach; e.g., the percentages in column seven show that when  $T \leq 0.4$ , the nominal optimal solution violates



the constraint on average 85% in the worst-case. When  $T \geq 0.715$  the nominal optimum solution no longer violates the constraint in the worst-case, but it is closer to be binding than the robust solution. All together, using our robust optimization method for this example, we gain immunity to the worst-case uncertainty without being penalized by the objective.

Now we analyze the average mean ( $\text{Avg}(\mathbb{E}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})])$ ) and variance ( $\text{Avg}(\text{Var}_{\mathbf{e}}[y(\mathbf{e}, \mathbf{d})])$ ) of the response for the nominal and robust optimal solutions, randomly sampling 1,000 probability vectors  $\mathbf{p}$  from the uncertainty set that is used in the RC; see (12). We sample as follows: First we uniformly sample 25 individual probabilities ( $\hat{p}_i$ ). Then to guarantee that the probabilities sum up to one, we use the adjustment:  $p_i = \hat{p}_i / \sum_{j=1}^{25} \hat{p}_j$ . Finally, if the probability vector  $\mathbf{p}$  is within the uncertainty set, we keep  $\mathbf{p}$  in the sample; else we discard  $\mathbf{p}$  and sample again. We repeat this procedure until we have 1,000 accepted vectors that are within the set. The confidence level of the uncertainty set (12) is 99.9%. From the  $\mathbf{p}$  in the sample we estimate the mean in (14) and the variance in (16) for a given solution, and then take the averages; see Table 7.

**Table 7** Average (Avg) Mean ( $\mathbb{E}$ ) and Variance (Var)

$T$	Robust		Nominal	
	Avg( $\mathbb{E}$ )	Avg(Var)	Avg( $\mathbb{E}$ )	Avg(Var)
0.1	35.37	0.064	35.41	0.12
0.2	35.40	0.095	35.45	0.24
0.3	35.40	0.095	35.46	0.35
0.4	35.40	0.095	35.47	0.46

Table 7 shows that the average expected response for the nominal solutions are less than 1% better than that of the robust solutions (we maximize  $\mathbb{E}(y)$ ; see (29)). Nevertheless, the average response variances of the robust optimal solutions are more than 100% better (smaller) than that of the nominal solutions when  $T \leq 0.2$ , and the improvement becomes more than 350% when  $T \geq 0.3$ . Moreover, we found that on average the nominal solutions violate 5% of the variance constraint, whereas the robust solutions are not at all binding. The effectiveness of our approach is further analyzed in the following subsection.

**5.1.2. Confidence Level** The uncertainty set (12) coincides with the  $(1 - \alpha)$ -confidence set for the unknown probability vector  $\mathbf{p}$  centered at the empirical estimate  $\mathbf{q}$ . In the numerical examples we use  $\alpha = 0.001$ , which means that the robust optimal solution is immune to uncertainty with at least 99.9% probability. These uncertainties are ignored in the nominal problem; nevertheless, the nominal solution may be robust for some part of the uncertainty set. To find the confidence level of a nominal solution, we calculate the tightest uncertainty set for which the nominal solution at hand is robust; i.e., we adjust the  $\rho$  value in (12); see Table 8.

**Table 8** Confidence Levels  $(1-\alpha)$  of Nominal Solutions

$T$	$\leq 0.4$	0.5	0.6	0.7	$\geq 0.8$
$(1-\alpha)$	0%	2%	70%	98%	99.9%

Table 8 shows that when  $T \leq 0.4$  the nominal solutions have no immunity to uncertainty at all; i.e., a ‘small’ change in the given empirical estimate  $\mathbf{q}$  results in infeasibility in the nominal variance constraint. We may anticipate this result, since the nominal solutions are binding for the associated cases. On the other hand, for  $T = 0.5$  to  $T = 0.7$  we see important improvement in the immunity to uncertainty for the nominal solution. In addition, when  $T \geq 0.8$ , the immunity of the nominal optimum solution is as good as the robust optimum solution—even though the nominal optimum solution is closer to be binding in the constraint than the robust solution. Concerning the objective, we have already shown in the last row of Table 6 that the robust and the ‘worst-case’ nominal optimum objective values are almost the same; i.e., the robust solutions are less than 1% better than the nominal solutions. In conclusion, the robust reformulation improves the immunity to uncertainty when  $T \leq 0.8$  and the improvement is even better when  $T \leq 0.4$ . Additionally, we prefer the robust solution when  $T \geq 0.8$ , since it performs better in the worst-case.

## 5.2. Distribution Center

In this example we focus on the simulation of a cross-docking distribution center (CDDC) developed by Shi (2011). The CDDC is used by a Chinese automobile manufacturer that needs to improve the physical flow in its supply chain. The objective of the company is steady production. The challenge in such production is the supply uncertainty due to environmental factors, mentioned below. To model the associated supply chain system, Shi proposes a hybrid approach that combines discrete-event simulation, RSM, and Taguchi’s world view. We focus on Shi’s linear regression metamodel for the total throughput with five controllable factors; namely, the number of receiving doors  $d_1$ , the number of shipping doors  $d_2$ , the number of forklifts  $d_3$ , the number of conveyors  $d_4$ , and the supply chain threshold in storage  $d_5$ . The model has two environmental factors; namely, the variability in less-than-truckload supply shipments  $e_1$ , and the production interruption or delay probabilities of the suppliers  $e_2$ . Using the simulation’s I/O data, Shi approximates the unknown I/O function of the total throughput by the following least-squares estimate  $\hat{y}$ :

$$\begin{aligned} \hat{y}_{\text{TT}}(\mathbf{e}, \mathbf{d}) = & 10^4 \times \left[ 47.97 + 3.982d_1 + 2.025d_2 - 0.031d_3 + 0.734d_4 + 0.034d_5 \right. \\ & + 0.789d_1d_2 + 0.012d_1d_3 + 0.003d_1d_4 - 0.002d_1d_5 + 0.0007d_2d_3 \\ & - 0.065d_2d_4 - 0.1131d_2d_5 - 0.078d_3d_4 - 0.041d_3d_5 + 0.11d_4d_5 \\ & \left. - 3.406d_1^2 - 1.781d_2^2 + 0.011d_3^2 - 1.033d_4^2 + 0.111d_5^2 \right] \end{aligned}$$

$$+ (16.66 + 1.511d_1 + 2.374d_2 - 0.059d_3 + 0.824d_4 - 0.093d_5)e_1 \\ - (0.005 + 0.27d_1 + 0.661d_2 - 0.086d_3 + 0.335d_4 - 0.005d_5)e_2],$$

where all factors are coded such that  $-1 \leq \mathbf{d} \leq 1$ , and  $-1 \leq \mathbf{e} \leq 1$ . More precisely, the coded controllable factors are between -1 and 1 because of the physical restrictions of the production facility. Shi's ANOVA shows that the metamodel  $\hat{y}_{\text{TT}}$  have non-significant lack-of-fit; and for the estimated parameters the level-of-significance is 0.05. Using Shi's response model, we focus on the robust reformulation of the following optimization problem:

$$\begin{aligned} \min_{-1 \leq \mathbf{d} \leq 1} \quad & \text{Var}_{\mathbf{e}}[\hat{y}(\mathbf{d}, \mathbf{e})] \\ \text{s.t.} \quad & E_{\mathbf{e}}[\hat{y}(\mathbf{d}, \mathbf{e})] \geq T. \end{aligned} \quad (37)$$

To estimate the frequencies  $\mathbf{q}$  used by the nominal and RC problems, we use the same historical data as in Figure 1.

**Table 9** Worst-Case Comparison for CDDC Example

$T$	Robust			Nominal (Worst-Case)		
	$\sigma_{\mathbf{e}}(\hat{y})$	$E_{\mathbf{e}}[\hat{y}]$	$E_{\mathbf{e}}[\hat{y}] \pm 3\hat{\sigma}_{\mathbf{e}}$	$\sigma_{\mathbf{e}}(\hat{y})$	$E_{\mathbf{e}}[\hat{y}]$	$E_{\mathbf{e}}[\hat{y}] \pm 3\hat{\sigma}_{\mathbf{e}}$
$\leq 339$	67	339	(136, 541)	67	339	(136, 541)
340	67	340	(137, 542)	67	339	(136, 541)
370	70	370	(158, 581)	68	348	(143, 552)
400	74	400	(176, 623)	71	377	(163, 590)
420	79	420	(182, 657)	73	396	(174, 617)
430	81	430	(184, 675)	75	405	(178, 632)
440	85	440	(184, 695)	77	414	(181, 648)
450	88	450	(183, 716)	80	424	(183, 664)

\*\*\* all entries should be multiplied by 1,000

Table 9 compares the worst-case performances of the nominal and robust optimal solutions. Besides the worst-case mean and standard deviation, we construct the worst-case confidence interval for  $\hat{y}$ ; namely,  $E_{\mathbf{e}}[\hat{y}] \pm 3\hat{\sigma}_{\mathbf{e}}$ . Because the probability distribution of the environmental factors is unknown, we cannot say much about the confidence level of this interval. The numerical results show that in the worst-case the nominal solutions are on average 6% lower than the target  $T$  for the expected total throughput; i.e., the average violation of the constraint is 6%. The worst-case standard deviations of the total throughput for the nominal solutions are on average 8% lower than that of the robust ones. Nevertheless the confidence intervals for the robust optimal solutions are always shifted to the right compared with the nominal solutions, which is in favor of the robust approach since a higher total throughput is better. Altogether the numerical results favor the robust approach when  $T > 339000$ ; when  $T \leq 339000$  both methods yield the same outcome.

**Table 10** Average Comparison for CDDC Example (Simulation Results)

$T$	Robust			Nominal		
	$\bar{\sigma}_e(\hat{y})$	$\bar{\mathbb{E}}_e[\hat{y}]$	$\bar{\mathbb{E}}_e[\hat{y}] \pm 3\bar{\sigma}_e$	$\bar{\sigma}_e(\hat{y})$	$\bar{\mathbb{E}}_e[\hat{y}]$	$\bar{\mathbb{E}}_e[\hat{y}] \pm 3\bar{\sigma}_e$
$\leq 339$	58	361	(186, 536)	58	360	(185, 535)
340	58	361	(186, 536)	58	360	(185, 535)
370	61	392	(209, 575)	59	369	(192, 547)
400	64	423	(229, 618)	61	399	(214, 585)
420	68	445	(238, 652)	64	419	(227, 612)
430	71	456	(241, 670)	65	429	(232, 627)
440	74	467	(244, 690)	67	439	(236, 643)
450	77	478	(245, 711)	69	449	(239, 659)

\*\*\* all entries should be multiplied by 1,000

In Table 10 we compare the average performance of robust and nominal solutions via Monte Carlo simulation. First, using the given historical data in Figure 1 as the nominal value ( $\mathbf{q}$ ) of the uncertainty set, we generate 1,000 probability vectors, like we did for the TV images example. Then we calculate the expected response and variance for the nominal and robust solutions at each probability vector, and take the averages. Table 10 shows higher means and lower variances than Table 9 in the worst-case. Table 9 is based on the worst-case scenario; i.e., we maximize the variance and minimize the expectation; Table 10 is based on the average performance. Table 10 shows that the confidence intervals for the robust solutions are always shifted to the right compared with the nominal solutions, so we prefer the robust approach, however, the robust solutions have slightly higher variations (from 0% to 10%) than the nominal solutions.

**Adjustable Robust Optimization** One of the most important decision factors in the CDDC is the number of shipping doors  $d_2$ . A moderate number of shipping doors may increase the inventory in the temporary storage area, whereas an excessive number causes a low utilization of doors. We now assume that  $d_2$  is adjustable according to the uncertain parameter  $e_1$ —namely, the variability in less-than-truckload supply shipments. More precisely, we assume that the number of shipping doors can be adjusted after the variability in supply shipments has been observed. We use the cell-based decision rule introduced in §4; the data and the 25 cells are presented in Figure 1. Notice that the domain of  $e_1$  is divided into five equal intervals; for each interval we have a different decision.

In Table 11, columns two through four present the actual ‘worst-case’ performance of the ARC, and columns five through eight present the average performance of adjustable robust solutions via Monte Carlo simulation as in Table 10. We use the same probability vector sample as in the television images example. These numerical results show that the worst-case confidence intervals of the ARC are tighter than those of the general RC in Table 9. This is because of the improved response

**Table 11** ARO Results for CDDC Example

$T$	Worst-Case			Average		
	$\sigma_e(\hat{y})$	$\mathbb{E}_e[\hat{y}]$	$\mathbb{E}_e[\hat{y}] \pm 3\sigma_e$	$\bar{\sigma}_e(\hat{y})$	$\bar{\mathbb{E}}_e[\hat{y}]$	$\bar{\mathbb{E}}_e[\hat{y}] \pm 3\bar{\sigma}_e$
$\leq 339$	67 (0%)	339	(136, 541)	58	367	(192, 541)
340	67 (0%)	340	(137, 542)	58	367	(193, 542)
370	68 (2%)	370	(167, 572)	59	389	(194, 543)
400	71 (3%)	400	(188, 611)	61	418	(212, 566)
420	73 (6%)	420	(201, 639)	63	425	(235, 615)
430	74 (7%)	430	(209, 650)	64	448	(246, 630)
440	75 (10%)	440	(215, 665)	65	449	(254, 644)
450	78 (10%)	450	(216, 684)	69	456	(249, 661)

\*\*\* all entries should be multiplied by 1,000

variances of the adjustable robust solutions, the improvement percentages (%) are reported in column two within brackets; e.g., it is as high as 10% when  $T=450000$ . As we anticipated, the simulation results show that the average performance of the adjustable robust solutions is better than the worst-case; i.e., the average of the response mean  $\bar{\mathbb{E}}_e[\hat{y}]$  is higher than the worst-case mean  $\mathbb{E}_e[\hat{y}]$ , and the average variance  $\bar{\sigma}_e(\hat{y})^2$  is lower than the worst-case variance  $\sigma_e(\hat{y})^2$ . Comparing the average performances of the non-adjustable and adjustable robust solutions in Tables 10 and 11 shows that the ARC yields tighter confidence intervals that are subintervals of the confidence intervals in the general RC. Therefore, the ARC reduces the response uncertainty compared with the general RC. Finally, additional experimentation showed that making ‘only’  $d_3$ ,  $d_4$  or  $d_5$  adjustable on  $e_1$  has an improvement less than 1% in the objective; i.e., in those cases the non-adjustable solutions are as good as the adjustable solutions. This shows that the quantitative value of information may significantly change for different parameters in ARO.

## 6. Conclusions and Future Research

In this paper, we proposed a novel methodology for RPD through RO. By definition, RPD distinguishes between decision and environmental variables. Our RO methodology uses an uncertainty set based on historical data on the environmental variables; this data may be collected from either real or simulated systems. Adding RO to RPD has the following advantages: (i) Unlike the classical RPD, we do not make any distributional assumptions on the uncertain parameters. (ii) We do account for the ambiguity caused by the lack of knowledge about distributions by using the so-called phi-divergence uncertainty sets based on historical data. (iii) Both RO and ARO methods are computationally tractable; ARO is tractable even for nonlinear decision rules. (iv) Our ARO approach can be used for modeling adjustable integer decision variables; to the best of our knowledge this is the first publication that models such variables.

In future research, we shall investigate our methodology for other metamodel types, such as higher-order polynomials, Kriging, and radial basis functions. We shall also apply our cell-based decision rules to general classes of optimization problems with limited number of ‘integer’ variables.

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